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On a Condition for a Graph to be a Tree

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ABSTRACT. In this paper we show that if a group G acts on the graph X under certain generators and relations of G, then X is a tree.

1. Introduction

The presentation of groups acting on trees known as Bass-Serre theorem has been given in[1], corollary 5.2.

The aim of this paper is to prove the converse of Bass-Serre theorem in the sense that if G is a group acting on a graph X and G has the presentation of corollary 5.2 of[1], then X is a tree.

We begin by giving some definitions. By a graph X we understand a pair of disjoint sets $V(X)$ and $E(X)$, with $V(X)$ non-empty, together with a mapping $E(X) \rightarrow V(X) x V(X), y \rightarrow (o(y), t(y))$, and a mapping $E(X) \rightarrow E(X), y \rightarrow \overline{y}$ satisfying $\overline{\overline{y}} = y$ and $o(\overline{y}) = t(y)$, for all $y \in E(X)$. The case $\overline{y} = y$ is possible for some $y \in E(X)$.

A path in a graph X is defined to be either a single vertex $v \in V(X)$ (a trivial path), or a finite sequence of edges $y_1, y_2, \cdots, y_n$, $n \geq 1$ such that $t(y_i) = o(y_{i+1})$ for $i = 1,2, \cdots, n-1$.

A path $y_1, y_2, \cdots y_n$ is reduced if $y_{i+1} \neq \overline{y}_i$, for $i = 1,2, \cdots, n-1$, A graph X is connected, if for every pair of vertices u and v of $V(X)$ there is a path $y_1, y_2, \cdots, y_n$ in X such that $o(y_1) = u$ and $t(y_n) = v$.

A graph X is called a tree if for every pair of vertices of $V(X)$ there is a unique reduced path in X joining them. A subgraph Y of a graph X consists of sets $V(Y) \subseteq V(X)$ and $E(Y) \subseteq E(X)$ such that if $y \in E(Y)$, then $\overline{y} \in E(Y)$, $o(y)$ and $t(y)$

are in $V(Y)$. We write $Y \subseteq X$. We take any vertex to be a subtree without edges. A maximal connected subgraph is called a component. It is clear that a graph is connected if and only if it has only one component.

If X_1 and X_2 are two graphs then the map $f: X_1 \rightarrow X_2$ is called a morphism if f takes vertices to vertices and edges to edges such that

$$f(\bar{y}) = \overline{f(y)}$$

$$f(o(y)) = o(f(y))$$

and $$f(t(y)) = t(f(y)), \qquad \text{for all } y \in E(X_1);$$

f is called an isomorphism if it is one-to-one and onto, and is called an automorphism if it is an isomorphism and $X_1 = X_2$. The automorphisms of X form a group under composition of maps, denoted by Aux (X).

We say that a group G acts on a graph X if there is a group homomorphism $\phi : G \rightarrow$ Aut (X). If $x \in X$ is a vertex or an edge, we write $g(x)$ for $\phi(g)(x)$. If $y \in E(X)$, then $g(\bar{y}) = \overline{g(y)}$, $g(o(y)) = o(g(y))$, and $g(t(y)) = t(g(y))$. The case $g(y) = \bar{y}$ for some $y \in E(X)$ and $g \in G$ may occur. If $y \in X$, (vertex or edge), we define $G(y) = \{g(y) \mid g \in G\}$ and this set is called an orbit. If $x, y \in X$, (vertices or edges) we define $G(x,y) = \{g \in G \mid g(y) = x\}$, and $G_x = G(x, x)$, called the stabilizer of x. For $y \in E(X)$, it is clear that G_y is a subgroup of G_u, where $u \in \{o(y), t(y)\}$. Also if Y is a subset of X then we define $G(Y)$ to be the set $G(Y) = \{g(y) \mid g \in G, y \in Y\}$.

It is clear that if $x \in V(X)$ and $y \in E(X)$, then $G(x,y) = \phi$.

For more details about groups acting on graphs we refer the reader to[1,2 or 3].

2. Preliminary Definitions and Notation

Throughout this paper G will be a group acting on the graph X, T a subtree of X such that T contains exactly one vertex from each G-vertex orbit, and Y a subtree of X such that Y contains T, and each edge of Y has at least one end in T, and Y contains exactly one edge y(say) from each G-edge orbit such that $G(\bar{y}, y) = \varphi$, and exactly one pair y and $\bar{y}$ from each G-edge orbit such that $G(\bar{y}, y) \neq \varphi$.

Properties of T and Y

(1) $G(Y) = X$.

(2) $G(V(T)) = V(X)$.

(3) If $u, v \in V(T)$ such that $G(u, v) \neq \varphi$, then $u = v$.

(4) $G(\bar{y}, y) = \varphi$, for all $y \in E(T)$.

(5) If $y_1, y_2, \in E(Y)$ such that $G(y_1, y_2) \neq \varphi$, then $y_1 = y_2$ or $y_1 = \bar{y}_2$

Given this we can now introduce the following notation.

(1) For each $v \in V(X)$ let v^* be the unique vertex of T such $G(v, v^*) \neq \phi$. In particular $v^* = v$ if $v \in V(T)$ and in general $(v^*)^* = v^*$. Also if $G(u, v) \neq \phi$, then $v^* = v^*$ for $u, v \in V(X)$. If $v \in V(T)$, let $< G_v \mid \text{rel}\, G_v >$ stand for any presentation of G_v, and $\widetilde{G}_v$ be the set of generating symbols of this presentation.

(2) For each edge y of $E(Y)$ we have the following

(a) Define $[y]$ to be an element of $G(t(y), t(y)^*)$, that is, $[y](t(y)^*) = t(y)$, to be chosen as follows.

If $o(y) \in V(T)$ then (i) $[y] = 1$ if $y \in E(T)$, (ii) $[y](y) = \bar{y}$ if $G(\bar{y}, y) \neq \phi$.

If $o(y) \notin V(T)$ then $[y] = [\bar{y}]^{-1}$ if $G(\bar{y}, y) = \varphi$, otherwise $[y] = [\bar{y}]$.

If is clear that $[y][\bar{y}] = 1$ if $G(\bar{y}, y) = \varphi$, otherwise $[y][\bar{y}] = [y]^2$.

(b) Let $-y = [y]^{-1}(y)$ if $o(y) \in V(T)$, otherwise let $-y = y$. Now define $+y = [y](-y)$.

It is clear that $t(-y) = t(y)^*$, $o(+y) = o(y)^*$ and $(+y) = -(\bar{y})$.

(c) Let S_y be a word in $G_{o(y)^*}$ of value $[y][\bar{y}$. It is clear that $S_{\bar{y}} = S_y$.

(d) Let E_y be a set of generators of G_{-y} and $\widetilde{G}_y$ be a set of words in $G_{t(y)^*}$ mapping onto E_y .

(e) Define $\phi_y : G_{-y} \to G_{+y}$ by $\phi_y(g) = [y]g[y]^{-1}$, $g \in G_{-y}$ and define $\psi_y : \widetilde{G}_y \to \widetilde{G}_{\bar{y}}$ by taking the word which represents the element g of E_y to the word which represents the element $[y]g[y]^{-1}$.

(f) Let $yG_y y^{-1} = G_{\bar{y}}$ stand for the set of relations $ywy^{-1} = \psi_y(w)$, $w \in \widetilde{G}_y$.

(3) Let $P(Y)$ stand for the set of generating symbols

(i) $\widetilde{G}_v$, for $v \in V(T)$

(ii) y , for $y \in E(Y)$

and $R(Y)$ stand for the set of relations

(i) $\text{rel}G_v$, for $v \in V(T)$

(ii) $yG_y y^{-1} = G_{\bar{y}}$, for $y \in E(Y)$

(iii) $y = 1$, for $y \in E(T)$

(iv) $y\bar{y} = S_y$, for $y \in E(Y)$

(v) $y^2 = S_y$, for $y \in E(Y)$ such that $G(\bar{y}, y) \neq \phi$.

Note that if $G(\bar{y}, y) \neq \phi$ then $y \notin E(T)$.

(4) Let $\delta(Y)$ be the set $\{ G_v, [y] : v \in V(T) \text{ and } y \in E(Y) \}$.

2.1 Theorem (Bass-Serre Theorem)

(i) If X is connected, then $\delta(Y)$ generates G.

(ii) If X is a tree, then G has the presentation $< P(Y) \mid R(Y) >$ via $\widetilde{G}_v \to G_v$ and $y \to [y]$, for all $v \in V(T)$ and all $y \in E(Y)$.

Proof

See[3], Corollary 5.2.

3. The Converse of Bass-Serre Theorem

Let G, X, Y and T be as in section two. In this section we prove the converse of Theorem 2.1 in the sense that if $\delta(Y)$ generates G, then X is connected, and if G has the presentation of Theorem 2.1 - (ii), then X is a tree.

3.1 Definition

For each $v \in V(Y)$ let X_v be an edge of $E(Y)$ such that $o(X_v) \in V(T)$ and $t(X_v) = v$. Let $e_v = 0$ if $v \in V(T)$, otherwise $e_v = 1$.

Concerning the edge X_v we see that X_v exists since Y is a subtree and X_v is unique if $v \notin V(T)$ and not necessarily unique if $v \in V(T)$.

The following proposition will be fundamental for the main theorem.

3.2 Proposition

Any element g of $G(u, v)$, where $u, v \in V(Y)$ can be written as $g = X_u]^{e_u} g_o[\bar{X}_v]^{e_v}$ where $g_o \in G_{u^*}$.

Proof

Since $g \in G(u, v)$, therefore $g(v) = u$.

We consider the following cases :

Case 1. u and v are in $V(T)$.

In this case we have $u^* = v^* = v$ so that $G(u, v) = G_v$ and, X_u and $\bar{X}_u$ are in $E(T)$ Since $[X_u] = [\bar{X}_u] = 1$ and $e_u = e_v = 0$, therefore the proposition holds.

Case 2. $u \in V(T)$ and $v \notin V(T)$.

In this case $u^* = v^* = u$, $[X_u] = 1$, $e_u = 0$, and $e_v = 1$.

$$\begin{aligned}
\text{Now } g \in G(u, v) &\Rightarrow g(v)\ u \\
&\Rightarrow g[X_v](v^*) = v^*, \text{ since } [X_v](v^*` = v, \text{ and } v^* = u \\
&\Rightarrow g[X_v] \in G_{v^*} \\
&\Rightarrow g[X_v] = h,\ h \in G_{v^*} \\
&\Rightarrow g = h[X_v]^{-1} \\
&\Rightarrow g = [X_u]^{e_u} h[X_v]^{-1}
\end{aligned}$$

If $G(\bar{X}_v, X_v` = \phi$, then $[X_v]^{-1} = [\bar{X}_v]$ We take $h = g_o$

If $G(\bar{X}_v, X_v) \neq \phi$, then $[\bar{X}_v] = [X_v]$ and $[X_v]^2 \in G_{x_v}$. Hence $[X_v]^{-1} = k[X_v]$, where $k \in G_{x_v}$. We take $g_o = hk$.

Case 3. $u \notin V(T)$ and $v \in V(T)$.

In this case $u^* = v^* = v$, $e_u = 1$, $e_v = 0$ and $[X_v] = 1$.

$$
\begin{aligned}
\text{Now } g \in G(u,v) &\Rightarrow g(v) = u \\
&\Rightarrow g(v) = [X_u](u^*) \\
&\Rightarrow g(v) = [X_u](v), \text{ since } u^* = v \\
&\Rightarrow [X_u]^{-1} g(v) = v \\
&\Rightarrow [X_u]^{-1} g \in G_v \\
&\Rightarrow [X_u]^{-1} g = g_o, \text{ for } g_o \in G_v \\
&\Rightarrow g = [X_u]^{e_u} g_o [\bar{X}_v]^{e_v}, \text{ since } e_u = 1, \text{ and } [X_v] = 1,
\end{aligned}
$$

Case 4. u and v are not in $V(T)$.

This case is similar to cases 2 and 3 above.

This completes the proof.

Since Y is a subtree of X, therefore any edge y of $E(Y)$, $o(y) \in V(T)$ can be written as $y = X_v$, where $v \in V(Y)$. Therefore by defining $e_y = e_v - 1$, where $v = o(y)$, for all $y \in E(Y)$, the following can be easily proved :

(1) $e_y + e_u = 0$ if $y \notin E(T)$, where $u = t(y)$

(2) $[y]^{e_u + e_v} = [y]$, where $u = t(y)$ and $v = o(y)$

(3) $[y]^{e_y + e_u} = [y]$, where $u = t(y)$

(4) $[X_u]^{e_u} = [y]^{e_u}$, where $u = t(y)$

(5) $[\bar{X}_v]^{e_v} = [y]^{e_v}$, where $v = o(y)$

3.3 Proposition

Let y_1 and y_2 be two edges of $E(Y)$, $u_i = t(y_i)$ and $v_i = o(y_i)$ for $i = 1, 2$ such that $G(u_1, v_2) \neq \phi$. Then any element $g \in G(u_1, v_2)$ can be written as

$$g = [y_1]^{e_{u_1}} g_o [y_2]^{e_{v_2}}, \text{ where } g_o \in G_{u_1^*}.$$

Proof

The proof easily follows from proposition 3.2 and (5) above.

3.4 Lemma

If G is generated by the set $\delta(Y)$, then X is connected.

Proof

Let C be a component of X such that C contains Y. We need to show that $X = C$.

Since $Y \subseteq C$, we have $G(Y) \subseteq G(C)$. By the definition of Y we have $G(Y) = X$. Therefore $G(C) = X$. To show that $C = X$ we need to show that $G_C = G$, where $G_C = \{ g \epsilon G \mid g(C) = C \}$ which is a subgroup of G. Define $\Delta(Y) = \{ g \epsilon G \mid Y \cap g(Y) \neq \phi \}$. Similarly $\Delta(C)$ is defined. Therefore $\Delta(Y) \subseteq \Delta(C)$.

Now we show that $\Delta(Y)$ generates G, *i.e.* $< \Delta(Y) > = G$. Since $\delta(Y)$ generates G, therefore we need to show that the elements of $\Delta(Y)$ can be written as a product of the elements of $\delta(Y)$.

Now $g \epsilon \Delta(Y) \Rightarrow Y \cap g(Y) \neq \phi$

$\Rightarrow$ there exists $u, v \epsilon V(Y)$ such that $u = g(v)$

$\Rightarrow g \epsilon G(u, v)$

$\Rightarrow g = [X_u]^{e_u} g_o [\bar{X}_v]^{e_v}$, where $g_o \epsilon G_{u^*}$. (Proposition 3.2)

$\Rightarrow < \delta(Y) > = < \Delta(Y) > = G$

$\Rightarrow < \Delta(C) > = G$

Since $< \Delta(C) > = G_C$, therefore $G_C = G$.

Therefore $G_C(C) = G(C)$, which implies that $C = X$. Hence X is connected.

This completes the proof.

To prove the main result of this paper we shall therefore assume the following condition on the elements of G.

Condition I

If $g_o[y_1] g_1 [y_2] g_2 \cdots [y_n] g_n$, $n \geq 1$ is the identity element of G, where

(1) $y_i \epsilon E(Y)$, for $1 \leq i \leq n$

(2) $t(y_i)^* = o(y_{i+1})^*$, for $1 \leq i \leq n-1$

(3) $g_o \epsilon G_{o(y_1)^*}$

(4) $g_i \epsilon G_{t(y_i)^*}$, for $1 \leq i \leq n$

then for some i, $1 \leq i \leq n$

(a) $y_{i+1} = \bar{y}_i$ and $g_i \epsilon G_{-y_i}$
or

(b) $y_{i+1} = y_i$ and $g_i \epsilon G_{y_i}$ if $G(\bar{y}_i , y_i) \neq \phi$.

The main result of this paper is the following theorem.

3.5 Theorem

If $\delta(Y)$ generates G, and G satisfies condition I, then X is a tree.

Proof

By Lemma 3.4, X is connected.

To show that X contains no circuits, that is, no reduced closed paths, we first show that X contains no loops. Suppose that x is a loop in X. Then $o(x) = t(x)$. Since $G(Y) = X$, $x = g(y)$ for $g \in G$ and $y \in E(Y)$ and so $g(o(g)) = g(t(y))$, hence $o(y) = t(y)$ contradicting the assumption that Y is a subtree. Hence X contains no loops.

Let $x_1, \cdots, x_n$, $n \geq 1$ be a close path in X. We need to show that this path is not a circuit, or equivalently, this path is not reduced. Now $o(x_1) = t(x_n)$ and $t(x_i) = o(x_{i+1})$ for $1 \leq i \leq n-1$. Since $G(Y) = X$, therefore, $x_i = g_i(y_i)$, for $g_i \in G$ and $y_i \in E(Y)$, $1 \leq i \leq n$. Let $u_i = t(y_i)$ and $v_i = o(y_i)$ for $1 \leq i \leq n$. From above we have $g_1(v_1) = g_n(u_n)$ and $g_i(u_i) = g_{i+1}(v_{i+1})$ for $1 \leq i \leq n-1$.

By proposition 3.3 we have $g_n^{-1} g_1 = [y_n]^{e_{u_n}} h_n [y_1]^{e_{v_1}}$ and $g_i^{-1} g_{i+1} = [y_i]^{e_{u_i}} h_i [y_{i+1}]^{e_{v_{i+1}}}$, where $h_i \in G_{u_i^*}$ for $1 \leq i \leq n-1$.

$$\begin{aligned} \text{Now } 1 &= g_1^{-1} g_2 g_2^{-1} \cdots g_{n-1} g_{n-1}^{-1} g_n g_n^{-1} g_1 \\ &= [y_1]^{\alpha_1} h_1 [y_2]^{\delta_2} [y_2]^{\alpha_2} h_2 \cdots [y_{n-1}]^{\alpha_{n-1}} h_{n-1} [y_n]^{\alpha_n} h_n [y_1]^{\delta_1} \end{aligned}$$

where $\alpha_i = e_{u_i}$ and $\delta_i = e_{v_i}$ for $1 \leq i \leq n$.

Conjugating the above equation by $[y_1]^{\delta_1}$ we get

$$\begin{aligned} &= [y_1]^{\gamma_1} h_1 [y_2]^{\gamma_2} h_2 \cdots y_{n-1}]^{\gamma_{n-1}} h_{n-1} [y_n]^{\gamma_n} h_n, \text{ where } \gamma_i = \delta_i + \alpha_i, 1 \leq i \leq n. \\ &= [y_1]\, h_1 [y_2] \cdots [y_{n-1}]\, h_{n-1} [y_n]\, h_n\,; \text{ since } [y_i]^{e_i} = [y_i], 1 \leq i \leq n, \text{ where } e_i = e_{y_i}. \end{aligned}$$

From condition I we have

(1) $y_{i+1} = \bar{y}_i$, and $h_i \in G_{-y_i}$, $1 \leq i \leq n-1$
or
(2) $y_{i+1} = y_i$, and $h_i \in G_{y_i}$, $1 \leq i \leq n-1$, where $G(\bar{y}_i, y_i) \neq \phi$.

If (1) holds then we have $[\bar{y}_{i+1}] = [y_i]$. We consider the following cases

$$\begin{aligned} \text{Case 1. } G(\bar{y}_i, y_i) &= \phi. \text{ Therefore we have} \\ g_i^{-1} g_{i+1} &= [y_i]^{\alpha_i} h_i [y_{i+1}]^{\alpha_i + 1} \\ & [y_i]^{\alpha_i} h_i [y_i]^{-\alpha_i}, \text{ since } \bar{y}_{i+1} = y_i \\ &= [y_i]^{e_i + \alpha_i} k_i [y_i]^{-e_j - \alpha_i}, \text{ where } k_i \in G_{y_i} \text{ such that} \\ & h_i = [y_i]^{e_i} k_i [y_i]^{-e_i} \\ &= k_i, \text{ since } [y]^{e_y + e_{i(y)}} = 1, \text{ for all } y \in E(Y). \end{aligned}$$

This implies that $g_i^{-1} g_{i+1} \in G_{y_i}$. That is,

$$\begin{aligned} g_i^{-1} g_{i+1}(y_i) &= y_i \\ \Rightarrow g_{i+1}(y_i) &= g_i(y_i) \\ \Rightarrow g_{i+1}(\bar{y}_{i+1}) &= g_i(y_i); \text{ since } y_{i+1} = \bar{y}_i \\ \rightarrow g_{i+1}(y_{i+1}) &= g_i(y_i) \end{aligned}$$

$\Rightarrow x_{i+1} \quad = \quad x_i$

$\Rightarrow$ the path $x_1, x_2, \cdots x_n$ is not reduced.

Case 2. $G(\bar{y}_i, y_i) \neq \phi$. Then $[y_i]^2 \in G_{y_i}$ and $\bar{y}_i] = [y_i] = [y_{i+1}$, since $y_{i+} = \bar{y}_i$

Therefore $g_i^{-1} g_{i+1} = [y_i]^{\alpha_i} h_i[y_i]^{\alpha_i}, h_i \in G_{y_i}$.

So $g_i^{-1} g_{i+1}(y_i) = [y_i]^{\alpha_i} h_i[y_i]^{\alpha_i}(y_i)$

$= [y_i]^{\alpha_i + e_i} k_i[y_i]^{\alpha_i - e_i}(y_i)$,

where $k_i \in G_{y_i}$ such that $h_i = [y_i]^{e_i} k_i[y_i]$

$$\begin{cases} k_i(y_i) & \text{if } t(y_i) \in V(T) \\ k_i[y_i]^2(y_i) & \text{if } t(y_i) \notin V(T). \end{cases}$$

Since k_i and $[y_i]^2$ are in G_{y_i}, therefore $k_i(y_i) = k_i[y_i]^2(y_i) = y_i$

Thus $g_i^{-1} g_{i+1}(y_i) = y_i$

$\Rightarrow g_{i+1}(y_i) = g_i(y_i)$

$\Rightarrow g_{i+1}(\bar{y}_{i+1}) = g_i(y_i)$, since $y_{i+1} = \bar{y}_i$

$\Rightarrow g_{i+1}(y_{i+1}) = g_i(y_i)$

$\Rightarrow \bar{x}_{i+1} = x_i$

$\Rightarrow$ the path $x_1, x_2, \cdots, x_n$ is not reduced.

Finally if (2) holds then we have $y_{i+1} = y_i$ and hence $[y_{i+1}] = [y_i]$.

Now $g_i^{-1} g_{i+1}(y_i) \quad [y_i]^{\alpha_i} h_i[y_i]^{\delta_i}(y_i)$

$y_i]^{\alpha_i + e_i} k_i[y_i]^{\delta_i - e_i}(y_i)$, where $k_i \in G_{y_i}$ such that

$h_i = [y_i]^{e_i} k_i[\ldots$

$k_i[y_i]^{\delta_i - e_i}(y_i)$, since $\alpha_i + e_i = 0$

$k_i[y_i](y_i)$, since δ_i

$= k_i(\bar{y}_i)$, since $[y \quad (y) = \bar{y}$ for all $y \in E(Y$

such that $G(\bar{y}, y) \neq \phi$

$= \bar{y}_i$, since $k_i \in G_{y_i}$ and $G_{\bar{y}} = G_y$ for all $y \in E(Y)$.

Hence $g_i^{-1} g_{i+1}(y_i) = \bar{y}_i$

$\Rightarrow g_{i+1}(y_i) = g_i(\bar{y}_i)$

$\Rightarrow g_{i+1}(y_{i+1}) = g_i(y_i)$, since $y_{i+1} = y_i$

$\Rightarrow x_{i+1} = \bar{x}_i$

$\Rightarrow$ the path $x_1, x_2, \cdots, x_n$ is not reduced.

This completes the proof of the main theorem.

We remark that if X is a tree then G satisfies condition I of Theorem 3.5, ([4], Corollary 1). In fact Corollary 1 of[2] has been proved in case $\delta(Y)$ generates G and G has the presentation $< P(Y) \mid R(Y)$ without using the assumption that X is a tree. This leads us to the following corollary of Theorem 3.5.

3.6 Corollary (The Converse of Bass-Serre Theorem)

If $\delta(Y)$ generates G, and G has the presentation $< P(Y) \mid R(Y) >$ via the map $\widetilde{G}_v \rightarrow G_v$ and $y \rightarrow [y]$ for all $v \in V(T)$ and all $y \in E(Y)$, then X is a tree.

4. Applications

In this section we give examples of groups acting on graphs and satisfying condition I of the main theorem. Free groups, free products of groups, free products of groups with amalgamation and HNN groups are examples of groups acting on trees in which condition I is the reduced form of the elements of these groups. For more details about the above groups we refer the reader to[4].

4.1 Free Groups

Let G be a group of base A.

Define the graph X as follows

$$V(X) = G$$

$$E(X) = Gx(A \cup A^{-1})$$

For $(g, a) \in E(X)$ we define

$$\overline{(g, a)} = (ga, a^{-1})$$

$$t(g, a) = ga$$

$$\text{and } o(g, a) = g$$

G acts on X as follows :

$$g(g') = gg' \text{ , for all } g, g' \in G$$

$$g(g', a) = (gg', a) \text{ for all } g, g' \in G \text{ and all } a \in A \cup A^{-1} .$$

It is clear that the stabilizer of each $g' \in G$ is trivial. We take $T = \{1\}$ and Y as $V(Y) = \{1\} \cup \{a \mid a \in A\}$, and $E(Y) = \{(1, a) \mid a \in A\} \cup \{(a, a^{-1}) \mid a \in A\}$. It is clear that Y is a subtree of X, $T \subseteq Y$ and $G(Y) = X$. Now we need to show that X is a tree. If u is a vertex of Y then $u^* = 1$ and if $a \in A$ then the edge $y = (1, a)$ is in Y, and, $o(y) = 1$, $t(y) = a$ and $[y] = a$. Therefore the set of $\delta(Y)$ of Lemma 3.4 is just the set $A \cup A^{-1}$ and the condition I is the reduced form of the elements of G. Consequently by Theorem 3.5, X is a tree.

4.2 Free Products of Groups

Let $G = *_{i \in I} G_i$, G_i non-trivial, $|I| > 1$, be a free product of the groups G_i

Define the graph X as follows :

$$V(X) = G \cup \{gG_i \mid g \in G, i \in I\}$$

$$E(X) = (GxI) \cup (IxG)$$

For $g \in G$ and $i \in I$ we define

$$\overline{(g, i)} = (i, g), \overline{(i, g)} = (g, i)$$

$$t(g, i) = gG_i, \; t(i, g) = g$$

$$\text{and } o(g, i) = g, \; o(i, g) = gG_i$$

We define the action of G on X by

$$g(g') = gg', \text{for all } g, g' \in G$$

$$g(g' G_i) = gg' G_i, \text{for all } g, g' \in G \text{ and all } i \in I$$

$$g(g', i) = (gg', i), \text{for all } g, g' \in G \text{ and all } i \in I$$

$$g(i, g') = (i, gg'), \text{for all } g, g' \in G \text{ and all } i \in I$$

Let T be defined as follows :

$$V(T) = \{1\} \cup \{G_i \mid i \in I\}$$

$$\text{and } E(T) = \{(1, i) \mid i \in I\} \cup \{i, 1) \mid i \in I\}$$

It is clear that T is a subtree of X, $Y = T$ and $G(Y) = X$. Therefore if v is a vertex of Y and y is an edge of Y then $v^* = v$ and $[y] = 1$. Also it is clear that the stabilizer of each edge is trivial, and the stabilizer of each vertex gG_i of X is the group G_i. Therefore the set $\delta(Y)$ of Lemma 3.4 is just the set $\cup \, i \in {}_I G_i$ and the condition I is the reduced form of the elements of G. Consequently by Theorem 3.5, X is a tree.

4.3 Free Products of Groups with Amalgamation

Let $G = *_A G_i$, $i \in I$, $|I| > 1$, A non-trivial, be a free product of the groups G_i with amalgamated subgroup A.

Define the graph X as follows :

$$V(X) = \{gA \mid g \in G\} \cup \{gG_i \mid g \in G_i\},$$

$$E(X) = \{(gA, i) \mid g \in G\} \cup \{i, gA) \mid g \in G\} \text{ such that}$$

$$\overline{(gA, i)} = (i, gA), (i, gA) = (gA, i)$$

$$o(gA, i) = gA, \; o(i, gA) = gG_i, \text{ and}$$

$$t(gA, i) = gG_i, \; t(i, gA) = gA.$$

We define the action of G on X by

$$g(g' A) = gg' A, \; g(g' G_i) = gg' G_i$$

$$g(g' A, i) = (gg' A, i) \text{ and } g(i, g' A) = (i \; gg' A$$

for all $g, g' \in G$ and $i \in I$.

Let T be defined as follows :

$V(T) = \{A, G_i \mid i \in I\}$ and $E(T) = \{(A, i), (i, A) \mid i \in I\}$ and $Y = T$.

It is clear that T is a subtree of X and $G(Y) = X$. If y is the edge (A, i) or (i, A), then $[y] = 1$.

It is clear that the stabilizer of the edge (A, i) is the group A and the stabilizer of the vertices A and G_i are the groups A and G_i respectively. Therefore the set $\delta(Y)$ of Lemma 3.4 is just the set $\cup_{i \in I} G_i$ and the condition I is the reduced form of the elements of G. Consequently by Theorem 3.5, X is a tree.

4.4 HNN Groups

Let $G = < H, t_i \mid \text{re} \mid H, t_i \; At_i^{-1} = B_i >$, $i \in I$ be HNN group of base H and associated subgroups A_i and B_i of H.

Define the graph X as follows :

$$V(X) = \{gH \mid g \in G\}$$

$$E(X) = \{(gB_i, t_i) \mid g \in G\} \cup \{gA_i, t_i^{-1} \mid g \in G\} \text{ such that}$$

$$(gB_i, t_i) = (gt_i A_i, t_i^{-1}), \; (gA_i, t_i^{-}) = (gt_i^{-1} B_i, t_i)$$

$$t(gB_i, t_i) = gt_i H, \; t(gAi, t_i^{-1}) = gt_i^{-1} H$$

and $o(gB_i, t_i) = gH, \; o(gA_i, t_i^{-1}) = gH.$

Let T and Y be defined as follows :

$T = \{H\}$, $V(V) = \{H\} \cup \{t_i H \mid i \in I\}$, and $E(Y) = \{(B_i, t_i) \mid i \in I_{\_} \cup \{(t_i A_i, t_i^{-1}) \mid i \in I\}$.

We define the action of G on X as follows :

$g(g' H) = gg' H$, $g(g' B_i, t_i) = (gg' B_i, t_i)$ and $g(g' A_i, t_i^{-1}) = (gg' A_i, t_i^{-1})$,

for all $g, g' \in G$.

It is clear that the stabilizer of the vertex gH is the group H, and the stabilizer of the edges (gB_i, t_i) and (gA_i, t_i^{-1}) are the groups B_i and A_i respectively.

Also Y is a subtree of X and $G(Y) = X$. If y is the edge (B_i, t_i) then $o(y) = H$, $t(y) = t_i H$, and $[y] = t_i$. Therefore the set $\delta(Y)$ of Lemma 3.4 is $H \cup \{t_i \mid i \in I\}$, and the condition I is the reduced form of the elements of G. Consequently by Theorem 3.5, X is a tree.

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عن أحـــد الشروط حتى يكون البـــيان شجـــرة

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المستخلص . في هذا البحث نبرهن على إنه إذا أثرت الزمرة G على البيان X تحت شرط معين على مولدات وعلاقات G فإن البيان X يكون شجرة .