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# ON CONVERGENCE THEOREM

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**Abstract** : The purpose of the present note is to establish a lattice property and a convergence pr generalized dominated-convergence theorem) of the extended map under hypotheses weaker than th and expressible in terms of order alone. In particular, "normality" is replaced by a separation proper weaker than the usual Hausdorff property.

## 1. INTRODUCTION

As Daniell showed [2], it is possible to develop the theory of the Lebesgue integral so as to essential role to the order properties. Recently this approach has been extended [3] so as to p theory of order-preserving maps which specializes to various integration process, spectral tions, etc.

As in Daniell's theory, one begins with an elementary integral or mapping I., defined on a s of some lattice $F$ (in Daniell's case, a real function space) and mapping $E$ monotonically into a p ordered set G. One then extends the domain of definition of the mapping so that the extended integral) I possesses desirable properties. Previous to introducing algebraic assumptions, t standingly important properties of I are lattice properties and convergence properties, the latt asserted in a generalized form of the Lebesgue dominated convergence theorem. The mappin defined, it is shown that algebraic properties of the elementary mapping, such as additivity or li are preserved in the extended mapping I.

However, in [3] the fundamental convergence theorem departs in two respects from the p thus sketched. First, the range-space G is assumed to be an additive group, even though no al properties are postulated for the mapping I. Thus one postulate of an algebraic nature intruc theory otherwise free of algebra. Second, $G$ is assumed to be "normal", which essestially ask be isomorphic with a subnet of a real function space. This is stronger than the requirement topologized in conformity with its order convergence, be a Hausdorff space; also, it brings in t number system in addition to the partially ordered sets $F$ and $G$.

**Definitions.** As in [3], we shall develop a "countable" and an "unrestricted" theory toge the device of brackets; in definitions, theorems, etc., either all bracketed expressions are to cluded, or else all are to be omitted.

If $G$ is partially ordered by an antisymmetric relation $\geq$, and $S \subset G$, and $b \in G$, then $b$ is an upper of $S$ if $g \in S$ imples $g \leq b$; and $b$ is the supremum $\vee S$ of $S$ if it is an upper bound of $S$, and for ever bound $b'$ of $S$ it is true that $b' \geq b$. Lower bounds and the infimum $\wedge S$ are defined dually.

The set G is Dedekind ($\sigma$-) complete if for every non-empty [countable] subset $S$ of $G$ wl deirected by $\geq$ and has an upper bound, the supremum $\vee$ S exists, and dually.

A function $f$ whose domain $D_f$ and range are both partially ordered is isotone if for all $x_1, x_2, \in$ I that $x_1 \leq x_2$, it is true that $f(x_1) \leq f(x_2)$; here we use $\leq$ as the symbol for the partial orderings i spaces. It is antitone if $x_1 \leq x_2$ implies $f(x_1) \geq f(x_2)$

If F is a distributive lattice and a, b, c are in F, one defines mid (a, b, c) to be $(a \vee b) \wedge (a \vee c) \wedge$
This is the same as $(a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$.
If F is a lattice under $\geq$, a partial ordering >> of F is a strengthening of $\geq$ if for all $f$, g, h, k in F

(a) $f >> g$ implies $f \geq g$,

(b) $f >> g$ and $g \geq k$ implies $f >> k$, and $f \geq g$ and $g >> k$ implies $f \geq k$.

(c) if $f >> h$ and $g >> k$, then $f \vee g >> h \vee k$ and $f \wedge g >> h \wedge k$.

A sequence $(\alpha_\upsilon : \nu = 1, 2, .....)$ of points of a partially ordered set G is order-convergent to a
of G if there exist subsets, P, Q of G with the following properties :

(a) P is directed by $\geq$, and $\vee P = a_0$.

(b) Q is directed by $\leq$, and $\wedge Q = a_0$.

(c) If $p \varepsilon P$ and $q \varepsilon Q$, then eventually (i.e., for all $\nu$ greater than some $\nu$) it is true that $p \leq a_\nu \leq q$.

This definition extends at once to nets $(a_\nu : \dot{\nu} \varepsilon N^*)$. $\succ$; we need only reinterpret "eventually" t
"for all $\nu \succ$ some $\nu \varepsilon N^*$". It extends similarly to syntaxes [4]. A syntax is a system $(f, R)$ in wl
a function and R is a filter-base in the domain of $f$. (A filter-base is a non-empty class of nonemp
dierected downwards by inclusion.) The definition of order-convergence can be applied also
taxes; we need only interpret "eventually" to mean "for all $\nu$ in a certain one $N$ of the class R o

## 1. MAIN RESULTS

The usual method of topologizing a partially ordered space is equivalent to the following. A
$V$ of a partially ordered set G is open if, and only if every syntax of points of G which conver
point of $V$ is eventually in $V$. (It would make no difference if we would replace "syntax" by "net"
definition.) A simple consequence is useful later.

**Corollary 2.1 :** If $S_1$ and $S_2$ are subsets of G directed by $\geq$ and $\leq$ respectively and having $\vee S_1$
and $V$ is an open subset of G containing $\vee S_1$, for some $g_1 \varepsilon S_1$ and $g_2 \varepsilon S_2$ the interval $[g_1$
contained in $V$.

**Proof :** The closed intervals $\{[g_1, g_2] : g_1 \varepsilon S_1, g_2 \varepsilon S_2\}$ form a filter-base R. Let id be the i
function, $I_\nu$, the definition of order-convergence take $P = S_1$, $Q = S_2$; then (id, R) converges to $\vee$
since $V$ is open it is eventually in $V$. That is, for all $\upsilon$ in some $[g_1, g_2] \varepsilon$ R the functional value i
is in $V$, which was to be proved.

If we apply this to the sets, P, Q in the definition of convergence, we obtain a corollary.

**Corollary 2.2 :** If a syntax $(f, R)$ converges to a point g in an open set V, the values of $f$ eventual
a closed interval contained in V.

The postulates involved in the definition of the extended map [3; 36] are the following.
Postulates. A [3] $[\sigma]$

(a) F is $\alpha$ $[\sigma]$ complete and infinitely distribute lattice under the partial ordering $\geq$.

(b) >> is a strengthening of $\geq$.

(c) G is a Dedekind complete partially ordered set, such that for each two elements $g_1$, $g_2$ of G, i
$g_2$ have an upper hound in G they also have a lower bound in G, and vice versa.

(d) $I_0$ is an isotone function whose domain is a subset E of F and whose range is contained

(e) For each pair $S_1$, $S_2$ of [countable] subsets of E directed by >>, << respectively and havi

(f) If $e_1$ and $e_2$ are in E, $I_0(e_1)$ and $I_0(e_2)$ have a common upper or lower bound in G, there exist è and e of E such that è << $e_i$ << e, i = 1, 2.

(g) If $e_1$, $e_2$ and $e_3$ are in E, and $I_0(e_1)$ and $I_0(e_2)$ have a common upper or lower bou then for every f in F such that f >> mid($e_1$, $e_2$, $e_3$) there is an e in E such that f >> e >> mid (e and dually.

A U-element is by definition an element u such that there exists a set S ⊂ E directed by having ∨S = u: each set is "associated" with u. If a set S associated with u has $I_0(S)$ boundd G, u is summable, and the image, or integral, $I_1(u)$ is defined to be ∨$I_0(S)$. *L*-elements and their are defined dually. If *f* is a U-element or an L-element, $I_1(f)$ is unique, and does not depen particular set S associated with *f* used in defining $I_1(f)$.

If *f* is in F, its lower integral I*f* is defined to be the supermum of $\ell_1(I)$ for all summable *L*-e $\ell \leq f$, and its upper integral I*f* is defined to be the infimum of $I_1(u)$ for all summable U-elem provided that such *U*-elements and *L*-elements exist. If I*f* = I*f*, *f* is summable, and their common denoted by I*f*.

Several rather simple properties of *U*-elements and L-elements are established in [3;37–47], we shall need the following.

**Theorem A [2.37-47] :** Let $u_1$, $u_2$, be summable *U*-elements such that $I_0(u_1)$ and $I_0(u_2)$ have a c upper or lower bound in G. Then $u_1 \vee u_2$ is *U*-element, and in the supremum of $\ell_1 \vee \ell_2$ for all su L-elements $\ell_1 \leq u_1$ and all summable L-elements $\ell_2 \leq u_2$; and if $\vee I_1(\ell_1 \vee \ell_2)$ exists for such $\ell_1$, $\ell_2$, i $\vee u_2$). Likewise $u_1 \wedge u_2$ is a summable *U*-element, and is the supremum of $\ell_1 \wedge \ell_2$ for the same $L_2$; and $\vee I_1(L_1 \wedge L_2) = I_1(u_1 \wedge u_2)$. The dual also holds, [2; Theorems 10.5, 10.6, 9.7.]

(ii) if $u_1$, $u_2$, ....... is an isotone sequence summable *U*-elements all ≤ α summable *U*-element u', u' is α summable *U*-element, and $I_1(\vee u_n) = \vee I_1(u_n)$.

Under the hypotheses of (i), it was easy to see that if $u_1$ and $u_2$ are summable so is $u_1 \wedge u_2$. But further postulates it cannot be shown that $u_1 \vee u_2$ is summable. For convenience in stating t postulate we introduce a new symbol. If A, B are subsets of a lattice F, by A [∨] B we shall m set (a ∨ b : a ε A and b ε B). If Д, Jb are families of subsets of F, by Д[V] Jb we shall mean the co of set (A [V] B : A ε π and B ε Jb). We shall be concerned with filter-bases Д, Jb of *L*-eleme members of sets of Д and of Jb being ≥ a fixed summable *L*-element *l**. If A εД and B ε Jb, by set A [V] B consists of summable *L*-elements and Д [V] Jb is a filter-base whose sets co summable L-elements ≥ *l*** All such *L*-elements are in the domain of $I_1$, so ($I_1$,Д), ($I_1$, Jb) and (I Jb) are all syntaxes of points of G. Our postulate is as follows.

**Postulate B :** Let *l** be a summable L-element, and let Д , Jb be filter-bases of sets of *L*-elemen *l**. Let each A εД and each B ε Jb be directed bath by ≥ and by ≤. Let the syntaxes ($I_1$,Д) and be convergent to points of G. Then (I1, Д[V] Jb) is also convergent to a point of G. The dual also

This is the case in particular when the postulates (12.1) of [3] hold, as follows at once from (12.(

**Lemma 2.2.** If (i) and postulate B hold, and $u_1$ and $u_2$ are summable $U$-elements such that $I_1$ $I_1(u_2)$ have a common lower or upper bound then $u_1 \vee u_2$ is a summable $U$-element.

**Proof :** (i), $u_1 \wedge u_2$ is summable, so there is a summable $L$-element $l^* \leq u_i$, $i = 1, 2$. Each sur $L$-element $l \leq u_1$ determines a set $A_1 = \{l' : l'$ a summable $L$-element, $l^* \vee l \leq l' \leq u_1\}$. Each $A_1$ is by $\geq$ and by $\leq$, and the collection $\Pi$ of such sets $A_1$ is a filter-base. Analogously, by using $l$ define sets $B_1$ and a filter-base Jb. Since $I_1(u_1) = \vee\{I_1(\ell) : \ell$ a summable L-element, $l \leq u_1\}$, th $(I_1, \Pi)$ converges to $I_1(u_1)$. Likewise $(I_1$, Jb) converges to $I_1(u_2)$. By Postulate B $(I_1, \Pi[V]$ Jb) co in G. Hence for some $A_1 \in \Pi$ and some B ..... in Jb, there exists a closed interval in G which c $I_1(l_1 \vee l_2)$ for all summable L-elements $l_1, l_2$ such that $l^* \vee l' \leq l_1 \leq u_1$ and $l^* \vee l'' \leq l_2 \leq u_2$. But t interval also contains the supremum of the set, which by (i) is $I_1(u_1 \vee u_2)$.

**Theorem 2.3.** Let (i) and postulate B be satisfied. If $f_1$, $f_2$ are summable, and $If_1$ and $If_2$ have a c upper or lower bound, $f_1 \vee f_2$ and $f_1 \wedge f_2$ are also summable.

**Proof :** By Lemma 2.2 we can find a summable U-element $u^*$ and a summable L-element $l^*$ s $l^* \leq f_i \leq u^*$, $i = 1, 2$. Given any summable U-element $u \geq f_1$ and any summable U-element I A [ Ì, ù] consist of all L-elements $\ell$ (necessarily summable) such that $\ell \vee \ell^* \leq \ell \leq u^* \wedge \grave{u}$. This is by $\leq$ and by $\geq$, and the collection $\Pi$ of all such sets is a filter-base. Analoguously, by use of sur L-elements $l \leq f_2$ and summable U-elements $\ddot{u} \geq f_2$ we define sets B[Ï, ü] and a filter-bases Jb. and $f_2$ are summable, it is easily seen that < $I_1$, U > and ($I_1$, Jb) converge to $If_1$ and $If_2$ respecti By Postulate B, < I1, $\Pi$[V] Jb > converges to some g k G. By the definition of convergence, thi that there exits subsets P, Q of G such that P is directed by $\geq$ and has $\vee P = g$, Q is directed b has $\wedge Q = g$, and for every $p \in P$ and $q \in Q$ the value of $I_1(\lambda)$ is in [p, q] for all $\lambda$ in some set A [ B [Ï, ü] of $\pi$ [V] Jb. But this set has a least member $\ell \grave{\vee} \ell \check{\vee} l^*$, which is a summable L-elements Hence $p \leq I(f_1, \vee f_2)$. On the other hand, the set has a supremum $\vee(l_1 \vee l_2 : \ell_1, \ell_2$ summable, $\ell_1 \leq$ $\ell_2 \leq u^* \wedge u_2$ and by (i) the value of $I_1(\ell_1 \vee \ell_2)$ for this set have $I_1([u_1, \wedge u^*] \vee [u_2 \wedge u^*])$ as sup This is a summable U-element $\geq f_1 \vee f_2$, whence $I(f_1 \vee f_2) \leq q$. If follows readily that $I(f_1, \vee F_2)$ a $f_2$) are both equal to g, and $f_1 \vee f_2$ is summable. Dually, so is $f_1 \wedge f_2$.

**Remark.** We have incidentally established that under the hypotheses of the Theorem, I ($f_1$ $\vee[I_1(\ell_1 \vee \ell_2) : \ell_1, \ell_2$ summable L-elements, $\ell_1 \leq f_1, \ell_2 \leq f_2]$.

In order to establish our first convergence theorem we find it desirable to assume a weakened the Hausdorff separation property in G.

**Postulate C :** For each pair a, b, of distinct points of G such that $a \leq b$, there exits disjoint ope $V_b$ which contain a and b respectively.

elements such that there exists a summable U-element u which satisfies $u \geq f_n$, $n = 1, 2, 3,$ $f_0 = \lim_{n \to \infty} f_n$. Then $f_0$ is summable, and $I(f_0) = \lim_{n \to \infty} I(f\infty)$.

**Proof :** Since the $I(f_n)$ rise but do not wxceed $I_1(u)$, they approach a limit $g_0$. Since $I(f_n) \leq I$ each n,

$$(*)\quad g_0 \leq I(f_0) \leq I(f_0)$$

It remains to establish

$$(**)\quad I(f_0) \leq g_0,$$

for then upper and lower integrals of $f_0$ will both be equal to $g_0 = \lim I f_n$.

Suppose (**) false. By (*) and (postulate C) there exist disjoint open sets U, V such that $g_0$ ε $\bar{I}(f_n)$ ε V. The set $U[\geq f_0]$ of all summable U-element $\geq f_0$ contains u, and the infimum of $I_1$ (U | $I(f_0)$. By (1.1), there exists $\bar{u}$ ε $U[\geq f_0]$ such that the interval $[I(f_0), I_1(u)]$ is contained in V.

We shall now define recursively a sequence of integers $j_1, j_2, \ldots\ldots$, a sequence of summable U-el $u_1, u_2, \ldots\ldots$ and a sequence of points $g_1, g_2, \ldots\ldots$ of the open set U having the following proper

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| (a) | $J_1$ | < | $j_2$ | < | $j_3$ | < | ........; | |
| (b) | $U_{k-1}$ | V | $f_{jk}$ | ≤ | $u_k$ | ≤ | u' | (k = 2, 3, .... |
| (c) | | | $I_1(u_k)$ | ε U | | | | (k = 1, 2, .... |
| (d) | $\lim_{n \to \infty} I(u_k \vee f_n) = g_k$ | | | | | | | (k = 1, 2, .... |

We describe the process of passing from stage (k – 1) to stage k; the first step in like this exc the symbol $u_0$ is to be replaced by $f_1$.

Since $\lim_{n \to \infty} I(_{k-1} \vee f_n) = g_{k-1}$ ε U, there exists a $j_k > j_{k-1}$ such that if $n \geq j_k$, $I(u_{k-1} \vee f_n)$ ε U. For sin of notation we denote $u_{k-1} \vee f_{ik}$ by the symbol $\varphi_k$. Since by Theorem 2.3 this is summable and $I($ by corollary 2.1 there exists a summable U-element $\ddot{u}_k$ such that for all. U-elements u satisfying $\leq \ddot{u}_k$ it is true that $I_1(u)$ ε U. We may suppose $\ddot{u}_k \leq \dot{u}$.

Give any summable L-element $l^*$ $\varphi_k$ and any summable U-element u* such that $\varphi_k \leq u^* \leq u_k''$, a non-empty class $L[l^*, u^*]$ of summable L-elements λ such that $l^* \leq \lambda \leq u^*$. These sets form base $Q^*$, and by the definition of the integral the syntax $(I_1, Q^*)$ converges to $I(\varphi_k)$.

Define $L_\infty$ to be the union of the sets $L[\leq U_{k-1} \vee f_n]$ of Summable L-element $\leq u_{k-1} \vee f_n$, $n = 1, 2, 3,$ each $l'$ in $L_\infty$, let $L_{l'}$ consist of all $l$ ε $L_\infty$ such that $l \geq l'$. These sets $L_{l'}$ ($l'$ ε $L_\infty$) also constitute base, which we call Q. Since $g_{k-1} = \vee n\, I(u_{k-1} \vee f_n) = \vee I_1(L_\infty)$ the syntax, $(I_1, Q)$ converges to postulate A $(I_1, Q^*\,[V]\,Q]$ also converges to some point of G. Since each set $L[l^*, u^*]\,[V]\,L_l$ filter-base contains the set $L_\lambda$, $\lambda = l' \vee l^*$ of the filter-base $Q$, $(I_1, Q)$ is a subsyntax of $(I1, Q^*$ [V the two have the same limit $g_{k-1}$. By Corollary 2.2, there exists a set $L[l^*, u^*]$ in $Q^*$ and a set $L_l$ in that all the values of $I_1(l_1 \vee l_2)$ with $l$ ε $L[\ell^*, u^*]$ and $l_2$ε$L_1$ lie in a closed interval $[\gamma_1, \gamma_2]$ containe Since $(I_1, Q^*)$ tends to $I(\varphi_k)$ ε U, we may suppose $I(u^*)$ ε U.

For some n, $u_{k-1} \vee f_n \geq l$, so $\lim_{n \to \infty} I(u^* \vee f_n) \geq I(u^* \vee f_n) \geq I(u^* \vee f_n) \geq I(1^* \vee l') \geq \gamma_1$. On the othe by the remark after Theorem 3.3, for all $n$ such that $f_n \geq l'$ we have $I(u^* \vee f_n) \leq \vee \{I_1(l_1 \vee l_2$ summable L-elements. $l_1 \leq u^*$, $l_2 \leq u_{k-1} \vee f_n) \leq \vee \{l_1 \vee l_2) : l_1$ ε $L[l^*, u^*]$, $l_2$ ε $L_1\} \leq \gamma_2$ Hence limn$_{\to}$ $\vee f_n)$ ε $[\gamma_1, \gamma_2]$. If we choose $u_k$ to be $u^*$ and $g_k$ to be $\lim_n i(u^* \vee f_n)$ we find that (a), (b), (c), (d) satisfied.

Now consider $u = V [u_k : k = 1, 2, 3, ......]$. This is a $U$-element, and (b) implies $f_k \leq u \leq u'$ (k =
Hence $u$ is a summable $U$-element such that $f_0 \leq u \leq u'$, whence $I_1(u)$ is contained in the close
$[I(f_0), I_1(u')]$, which is already known to be contained in $V$. Since by (ii), $I_1(u) = V [I_1(u_k) : k = 1,$
follows that for all large k, $I_1(u_k)$ is also in V. But by (c), $I_1(u_k)$ is also in U; and U and V are disj
contradiction establishes the theorem.

From Postulate C it is easy to deduce a generalized form of the Lebesgue dominated-con
theorem, as follows :

**Theorem 2.4 :** Let postulates (A), (B) and (C) hold. Let $\grave{f}, \ddot{f}, f_1, f_2, ......$ be summable elements
$\grave{f} \leq f_n \leq \ddot{f}$, n = 1, 2, 3, ..... If $\lim_{n \to \infty} f_n$ exists, it is summable, and $I(\lim_{n \to \infty} f_n) = \lim_{n \to \infty} I(f_n)$.
By Postulate C and its dual, the elements $\varphi_k = \wedge [f_n : n = k, k + 1, ...] = \lim_{j \to \infty} \wedge [f_k, f_{k+1}, .... F_{k}$
$= V \{f_n : n = k, k+1, ....\} = \lim_{j \to \infty} V \{f_k, f_{k+1}, ... F_{k+j}\}$ are summable; the sequence $\varphi_1, \varphi_2, ...$ is iso
the sequence $\psi_1, \psi_2, .....$ is antitone, and both converge to $\lim f_n$, so by postulate C this is su
Also, the set $P = \{I(\varphi_k) : k = 1, 2, .....\}$ is directed by $\geq$ and has supremum $I(\lim f_n)$ in G, and th
$\{I(\psi_k) : k = 1, 2, ....\}$ is directed by $\leq$ and has infimum $I(\lim f_n)$, and for $n \geq k$ we have $I(\varphi_k) \leq I(f_n$
so $\lim I(f_n)$ is by definition $I(\lim f_n)$.

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