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Positive solution for fourth order boundary value problem

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ABSTRACT

In this paper, we investigate the problem of existence of positive solutions for the nonlinear fourth order boundary value problem:

$$D^4u(t) = \lambda a(t)f(u(t)), \quad 0 < t < 1,$$
$$u(0) = u''(0) = u'(1) = u'''(1) = 0,$$

where λ is a positive parameter. By using Krasnoesel'skii's fixed point theorem of cone, we establish various results on the existence of positive solutions of the boundary value problem. Under various assumptions on $a(t)$ and $f(u(t))$, we give the intervals of the parameter λ which yield the existence of the positive solutions.

Key words: Fourth order boundary value problem, Krasnoesel'skii's fixed point theorem, Green's function, Positive solution.

INTRODUCTION

One of the most frequently used tools for proving the existence of positive solutions to the integral equations and boundary value problems is Krasnoesel'skii's theorem on cone expansion and compression and its norm-type version due to Guo and Lakshmikantham[11]. Very recently, Li[12] used Krasnoesel'skii's fixed point theorem to prove some existence results to the nonlinear third singular boundary value problem:

$$u'''(t) + \lambda a(t)f(u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = u''(1) = 0.$$

In fact, the problem of existence of positive solutions for the nonlinear fourth order boundary value problem has been studied by many authors under various boundary conditions and by different approaches. For some other results, We refer the reader to papers of Agarwal[1], Agarwal, O'Regan and Wong[2], Avery, Chyan and Henderson[3], Chyan and Henderson[7], Dalmasso[8] and Guo and Lakshmikantham[11]. The purpose of this paper is to establish the existence of positive solutions to nonlinear fourth order boundary value problem:

$$D^4u(t) = \lambda a(t)f(u(t)), \quad 0 < t < 1, \qquad (1)$$
$$u(0) = u''(0) = u'(1) = u'''(1) = 0, \qquad (2)$$

where λ is a positive parameter. Throughout the paper, we assume that

C1: $f : [0,\infty) \to [0,\infty)$ is continuous

C2: $a : (0,1) \to [0,\infty)$ is continuous function such that $\int_0^1 a(t)dt > 0$.

Here, by a positive solution of the boundary value problem we mean a function which is positive on $(0,1)$ and satisfies differential equation (1) and the boundary condition (2).

Definition 1. *Let E be a real Banach space. A nonempty closed set* $K \subset E$ *is called* ***cone*** *of E if it satisfies the following conditions:*

1. $x \in K$, $\lambda \geq 0$ implies that $\lambda x \in K$
2. $x \in K$, $-x \in K$ implies that $x = 0$.

Definition 2. *An operator is called* ***completely continuous*** *if it is continuous and maps bounded sets into precompact sets.*

The Green's function $G : [0,1] \times [0,1] \to [0, \infty)$ for the problem (1) – (2) is

$$G(t,s) = \begin{cases} \dfrac{(1-(1-s)^2)}{2} t - \dfrac{t^3}{6}, & \text{if } 0 \leq t \leq s \leq 1; \\ \dfrac{((1-s)^2 - 1)}{2} t - \dfrac{t^3}{6} + \dfrac{(t-s)^3}{6}, & \text{if } 0 \leq s \leq t \leq 1. \end{cases}$$

Then problem (1) – (2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) a(s) f(u(s))\, ds, \quad 0 \leq t \leq 1. \quad (3)$$

It is easy to verify that G is a continuous function, and $G(t,s) \geq 0$ if $t,s \in (0,1)$. We will need the following fixed point theorem, which is due to Krasnosel'skii[10], to prove some of our results.

Theorem 1. *Let E be a Banach space and* $K \subset E$ *is a cone in E. Assume that* Ω_1 *and* Ω_2 *are open subsets of E with* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$. *Let* $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ *be completely continuous operator. In addition suppose either:*

H1: $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ $\forall u \in K \cap \partial\Omega_2$ or,

H2: $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_2$ and $\|Tu\| \geq \|u\|$ $\forall u \in K \cap \partial\Omega_1$, *holds. Then T has a fixed pint in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

SOLUTION IN THE CONE

In this section, we will apply Krasnoesel'skii's fixed point theorem to the eigenvalue problem (1) – (2). We note that $u(t)$ is a solution of (1) – (2) if and only if

$$u(t) = \lambda \int_0^1 G(t,s) a(s) f(u(s))\, ds, \quad 0 \leq t \leq 1.$$

Now, we consider the Banach space $E = C[0,1]$ equipped with standard norm

$$\|u\| = \max_{0 \leq t \leq 1} |u(t)|, \quad u \in E.$$

We define a cone P by

$$P = \left\{ u \in E : \min_{0 \leq t \leq 1} u(t) \geq \frac{m}{M} \|u\| \right\},$$

where $m = \min_{0 \leq s \leq 1} \{G(t,s), 0 \leq t \leq 1\}$ and $M = \max_{0 \leq s \leq 1} \{G(t,s), 0 \leq t \leq 1\}$. It is easy to see that if $u \in P$, then $\|u\| = u(1)$. Define an integral operator $T : P \to E$ by

$$Tu(t) = \lambda \int_0^1 G(t,s) a(s) f(u(s))\, ds.$$

It is clear that if (C1) and (C2) hold, then $T : P \to P$ is a completely continuous operator. Following Sun and Wen[13], we define some important constants:

$$F_0 = \limsup_{u \to 0^+} \frac{f(u)}{u} \qquad f_0 = \liminf_{u \to 0^+} \frac{f(u)}{u}$$

$$F_\infty = \limsup_{u \to +\infty} \frac{f(u)}{u} \qquad f_\infty = \liminf_{u \to +\infty} \frac{f(u)}{u}$$

Here we assume that $\dfrac{1}{MAf_\infty} = 0$, if $f_\infty \to \infty$, $\dfrac{1}{mAF_0} = \infty$, if $F_0 \to 0$, $\dfrac{1}{MAf_0} = 0$, if $f_0 \to \infty$ and $\dfrac{1}{mAF_\infty} = \infty$, if $F_\infty \to 0$. We assume that $A = \int_0^1 a(s) ds$.

Lemma 1. $T(P) \subset P$.

Proof. Since $G(t,s) \geq 0$ and

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s) a(s) f(u(s))\, ds \\ &\geq \lambda m \int_0^1 a(s) f(u(s))\, ds \\ &\geq \lambda m \int_0^1 \frac{G(t,s)}{M} a(s) f(u(s))\, ds \geq \frac{m}{M} Tu(t), \end{aligned}$$

for all $t,s \in [0,1]$.

Theorem 5. *Suppose that* $\lambda mAf(u) > u$ *for* $u \in (0,\infty)$. *Then the problem (1) and (2) has no positive solution.*

Proof. Assume to the contrary that u is a positive solution of (1) and (2). Then

$$u(1) = \lambda \int_0^1 G(1,s)a(s)f(u(s))\,ds$$
$$> \frac{1}{mA}\int_0^1 G(1,s)a(s)u(s)\,ds$$
$$\geq \frac{\|u\|}{A}\int_0^1 a(s)\,ds = u(1).$$

This is a contradiction and completes the proof.

Example 1. Consider the boundary value problem

$$D^4u(t) = \lambda(\frac{9}{2}t^2)\frac{(u(t)+9u^2(t))}{(1+u(t))}(8+\cos u), \quad (4)$$

$$u(0) = u''(0) = u'(1) = u'''(1) = 0, \quad (5)$$

for $0 \leq t \leq 1$, where $\lambda > 0$ is a parameter, together with the boundary conditions (5). We consider $a(t) = \frac{9}{2}t^2$ and

$$f(u) = \frac{(u+9u^2)}{(1+u)}(8+\cos u).$$

Then $F_0 = f_0 = 9$, $F_\infty = 72$, $f_\infty = 63$ and $9u < f(u) < 72u$. We assume that $m = 0.123$ and $M = 0.333$. From theorem 2, we see that if $\lambda \in (0.0143, 0.037)$, then the problem (4), (5) has a positive solution. By theorem 4, we have that if $\lambda < 6.5$, then the problem (4), (5) has no positive solution. By theorem 5, we have that if $\lambda > 2.34$, then the problem (4), (5) has no positive solution.

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